A STUDY OF α-VARIATION. I.

BY
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This paper is based on the notion of the higher variation of a function introduced by N. Wiener [9] while studying the Fourier coefficients of a function with bounded variation. L. C. Young applied this idea to derive a new existence theorem for Stieltjes integration and later collaborated with E. R. Love in publishing a number of papers on subjects related to this concept.

PRELIMINARIES

1.1. Suppose that $f(x)$ is a real- or complex-valued function defined over $a \leqq x \leqq b$. For $0 < \alpha \leqq 1$, we define the α-variation of $f(x)$ over this interval as the least upper bound of the sums

$$\left\{ \sum_{n=1}^{N} | f(x_n) - f(x_{n-1}) |^{1/\alpha} \right\}^{\alpha}$$

taken over all subdivisions $a = x_0 < \cdots < x_N = b$, and we denote this upper bound by

$$V_\alpha\{f(x);\, a \leqq x \leqq b\} \quad \text{or} \quad V_\alpha\{f(x);\, x \in I\},$$

where I is the interval $a \leqq x \leqq b$. Similarly we define the 0-variation (or oscillation) of $f(x)$ over this interval as the least upper bound of the difference $|f(x'') - f(x')|$ for $a \leqq x' < x'' \leqq b$, and we denote this upper bound by

$$V_0\{f(x);\, a \leqq x \leqq b\} \quad \text{or} \quad V_0\{f(x);\, x \in I\}.$$

It is often convenient to consider the α-variation of a function over an interval, I, which is open or half-open, and we can appropriately define the symbol

$$V_\alpha\{f(x);\, x \in I\}$$

for $0 \leqq \alpha \leqq 1$.

Suppose that $\{x_n\}$ is any set of real or complex numbers. For $0 < \alpha \leqq 1$, we denote by

$$\left\{ \sum_{n} | \sum x_n |^{1/\alpha} \right\}^{\alpha}$$

the least upper bound of all sums

Presented to the Society, April 25, 1953; received by the editors March 23, 1953.

$$\left\{ \sum_k | x_{n_{k-1}} + \cdots + x_{n_k - 1} |^{1/\alpha} \right\}^\alpha,$$

where $\{n_k\}$ is any appropriate finite sequence. When $\alpha = 0$, we let

$$\left\{ \sum_n | x_n |^{1/\alpha} \right\}^\alpha \quad \text{and} \quad \left\{ \sum_n | \sum x_n |^{1/\alpha} \right\}^\alpha$$

denote the least upper bounds for $|x_n|$ and $|x_m + \cdots + x_n|$ respectively.

1.2. For $0 \leqq \alpha \leqq 1$, we say that the function $f(x)$ is in W_α over the interval $0 \leqq x \leqq 1$ if $f(x)$ has bounded α-variation over this interval; W_0 is simply the class of bounded functions. The sum of two functions in W_α is also in W_α since, by Minkowski's inequality,

$$V_\alpha\{f(x) + g(x); x \in I\} \leqq V_\alpha\{f(x); x \in I\} + V_\alpha\{g(x); x \in I\}.$$

Similarly, for $0 \leqq \alpha < \beta \leqq 1$, it follows from Jensen's inequality that

$$V_\alpha\{f(x); x \in I\} \leqq V_\beta\{f(x); x \in I\},$$

and hence a function in W_β is also in W_α.

If, for $0 \leqq \alpha \leqq 1$, $f(x)$ has period 1 and is in the class Lip (α), i.e. if for some constant C

$$| f(x'') - f(x') | < C(x'' - x')^\alpha$$

for each $x' < x''$, $f(x)$ is obviously in W_α and the α-variation of $f(x)$ over any interval of length 1 is less than C. The converse is not true since functions in W_α need not be continuous. However we can prove the following result.

THEOREM 1.2.1. (*Cf.* [11, *p.* 455].) *Suppose that* $0 \leqq \alpha \leqq 1$, *that* $f(x)$ *is real and continuous with period* 1, *and that the α-variation of* $f(x)$ *over any interval of length* 1 *is less than* 1. *There exists a continuous increasing function* $\phi(t)$ *such that* $\phi(t+1) = \phi(t) + 1$ *and such that*

$$| f\{\phi(t'')\} - f\{\phi(t')\} | < (t'' - t')^\alpha$$

for each $t' < t''$.

We can suppose that $\alpha > 0$ and that $f(x)$ assumes its positive maximum at $x = 0$. Define the function $\gamma(x)$ so that $\{\gamma(x)\}^\alpha$ is equal to the α-variation of $f(x)$ over the interval $0 \leqq y \leqq x$. $\gamma(x)$ is continuous, $\gamma(x+1) \geqq \gamma(x) + \gamma(1)$, and $\gamma(1) < 1$. For each $x > 0$ we can select a subdivision $0 = y_0 < \cdots < y_N = x + 1$ such that

$$\gamma(x + 1) < \sum_{n=1}^N | f(y_n) - f(y_{n-1}) |^{1/\alpha} + \epsilon.$$

If $y_n < 1 \leqq y_{n+1}$, we see by periodicity that

$$|f(y_{n+1}) - f(y_n)|^{1/\alpha} \leqq |f(y_{n+1}) - f(1)|^{1/\alpha} + |f(1) - f(y_n)|^{1/\alpha},$$

and hence that $\gamma(x+1)<\gamma(x)+\gamma(1)+\epsilon$. For $x>0$ we conclude that $\gamma(x+1) =\gamma(x)+\gamma(1)$ and we extend $\gamma(x)$ so this holds for all x.

Let $\theta(x)=\gamma(x)+x\{1-\gamma(1)\}$. $\theta(x)$ is continuous and increasing for all x; let $\phi(t)$ be the inverse function. Then $\phi(t+1)=\phi(t)+1$ and

$$|f\{\phi(t'')\} - f\{\phi(t')\}|^{1/\alpha} \leqq \gamma\{\phi(t'')\} - \gamma\{\phi(t')\} < \theta\{\phi(t'')\} - \theta\{\phi(t')\}$$

for each $t'<t''$. This completes the proof.

We also have the following

THEOREM 1.2.2. *Suppose that* $0\leqq\alpha\leqq 1$ *and that* $f(x)$ *is in* W_α. *Then*

$$V_\alpha\{f(y);\, x \leqq y \leqq x+h\} = O(h^\alpha)$$

almost everywhere, and similarly on the left.

Assume that $\alpha>0$ and let E_k be the set of points in $0\leqq x<1$ for which

$$\limsup_{h\to 0} h^{-\alpha} V_\alpha\{f(y);\, x \leqq y \leqq x+h\} > k.$$

For each x in E_k and each $\epsilon>0$ there exists an interval, $I(x)\equiv x\leqq y\leqq x+h$, such that

$$V_\alpha\{f(y);\, x \leqq y \leqq x+h\} \geqq kh^\alpha$$

and such that $h<\epsilon$. By Vitali's covering theorem there exists a sequence, $\{I(x_n)\}$, of nonoverlapping intervals which cover almost all of E_k and

$$\begin{aligned}\{\text{outer meas } E_k\}^\alpha &\leqq \left\{\sum_{n=1}^{\infty} |I(x_n)|\right\}^\alpha \\ &\leqq \frac{1}{k}\cdot\left\{\sum_{n=1}^{\infty} V_\alpha\{f(y);\, y\in I(x_n)\}^{1/\alpha}\right\}^\alpha \\ &\leqq \frac{1}{k}\cdot V_\alpha\{f(x);\, 0\leqq x\leqq 1\}.\end{aligned}$$

The set of points in $0\leqq x<1$ for which

$$V_\alpha\{f(y);\, x \leqq y \leqq x+h\} \neq O(h^\alpha)$$

is contained in E_k for each k and must have zero measure.

COROLLARY 1.2.3. *Suppose that* $0\leqq\alpha\leqq 1$ *and that* $f(x)$ *is in* W_α. *Then*

$$|f(x+h) - f(x)| = O(h^\alpha)$$

almost everywhere, and similarly on the left.

Hence a function in W_α satisfies a Lip (α) condition almost everywhere but of course not uniformly.

1.3. We need the following lemma in order to study the relation between W_α and the Hardy-Littlewood integrated Lipschitz classes [4, p. 612].

LEMMA 1.3.1. *Suppose that* $0 \leqq \alpha \leqq 1$, *that* $f(x)$ *is a real-valued function with period* 1, *and that the* α-*variation of* $f(x)$ *over any interval of length* 1 *does not exceed* 1. *Then the* α-*variation of* $f(x)$ *over any interval of length* k *does not exceed* k^α *for each positive integer* k.

We need only show for $0 < \alpha \leqq 1$ that

1.3.2 $$V_\alpha\{f(x);\, 0 \leqq x \leqq k\} \leqq k^\alpha.$$

Pick a subdivision σ, $0 = x_0 < \cdots < x_N = k$, so that the left-hand side of 1.3.2 is majorized by

$$\left\{\sum_\sigma |\Delta f|^{1/\alpha}\right\}^\alpha + \epsilon = S^\alpha + \epsilon.$$

We can assume that σ contains two points, c and d, where $d = c+1$ and where $f(d) = f(c) = \mathrm{Max}_n f(x_n)$ for adding such points to σ does not decrease S. (Cf. proof for 1.2.1.) We have

$$\begin{aligned} S &= \sum_{\sigma \cap [0,c]} |\Delta f|^{1/\alpha} + \sum_{\sigma \cap [c,d]} |\Delta f|^{1/\alpha} + \sum_{\sigma \cap [d,k]} |\Delta f|^{1/\alpha} \\ &= S_1 + S_2 + S_3. \end{aligned}$$

S_2 is majorized by 1, $\{S_1 + S_3\}^\alpha$ is majorized by the α-variation of $f(x)$ over $0 \leqq x \leqq k-1$, and 1.3.2 follows by induction.

THEOREM 1.3.3. (*Cf.* [10, *p.* 259].) *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $f(x)$ *is a measurable real-valued function with period* 1. *If the* α-*variation of* $f(x)$ *over any interval of length* 1 *never exceeds* 1, *then*

$$\left\{\int_0^1 |f(x+h) - f(x)|^{1/\alpha} dx\right\}^\alpha \leqq h^\alpha$$

for every $h > 0$.

Assume that $\alpha > 0$ and let $h = m/n$, where m and n are relatively prime positive integers. Then

$$I(h) = \int_0^1 |f(x+h) - f(x)|^{1/\alpha} dx = \sum_{\nu=1}^n \int_{(\nu-1)/n}^{\nu/n} \left| f\left(x + \frac{m}{n}\right) - f(x)\right|^{1/\alpha} dx$$

and, because $f(x)$ has period 1, this last sum is equal to

$$\int_0^{1/n} \left\{ \sum_{\nu=1}^n \left| f\left(x + \frac{\nu m}{n}\right) - f\left(x + \frac{(\nu-1)m}{n}\right)\right|^{1/\alpha}\right\} dx \leqq \int_0^{1/n} m\,dx = h.$$

Hence the theorem is true for rational h. Since $f(x)$ is bounded, $I(h)$ is continuous and the theorem holds for all h.

COROLLARY 1.3.4. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $f(x)$ *is a measurable real-valued function with period* 1. *Then*

$$\left\{\int_0^1 |f(x+h) - f(x)|^{1/\alpha} dx\right\}^\alpha \leqq 2^\alpha V_\alpha\{f(x); 0 \leqq x \leqq 1\} h^\alpha$$

for every $h > 0$.

This follows from 1.3.3 since the α-variation of $f(x)$ over any interval of length 1 is majorized by

$$\{V_\alpha\{f(x); 0 \leqq x \leqq 1\}^{1/\alpha} + V_0\{f(x); 0 \leqq x \leqq 1\}^{1/\alpha}\}^\alpha.$$

If $f(x) = e^{2\pi i x}$ and $\alpha = 1/2$, the α-variation of $f(x)$ over any interval of length 2 exceeds 2^α times the α-variation of $f(x)$ over any interval of length 1. Since 1.2.1 implies 1.3.1 when $f(x)$ is continuous, the restriction that $f(x)$ be real is essential in both of these results. The same is true for 1.3.3.

From 1.3.4 it is obvious that any measurable function with bounded α-variation over some interval, $a \leqq x \leqq b$, is in the Hardy-Littlewood class Lip $(\alpha, 1/\alpha)$ over that interval. The converse is not true. Hardy and Littlewood [4, p. 621] point out that if

$$f(x) = \log \frac{1}{|x|}, \qquad x \neq 0,$$

then, for $h > 0$,

$$\left\{\int_{-\pi}^{\pi} |f(x+h) - f(x)|^{1/\alpha} dx\right\}^\alpha = O(h^\alpha), \qquad 0 < \alpha < 1,$$

while $f(x)$ is not even bounded in the neighborhood of $x = 0$.

1.4. The following lemma generalizes a familiar theorem on uniform continuity.

LEMMA 1.4.1. *Suppose that* $0 \leqq \alpha \leqq 1$, *that* $f(x)$ *has bounded* α*-variation over* $0 \leqq x \leqq 1$, *and that* $f(x)$ *is continuous in this interval. For* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that, for* $0 \leqq x_0 < x_0 + \delta \leqq 1$, *we have*

$$V_\alpha\{f(x); x_0 \leqq x \leqq x_0 + \delta\} < \epsilon.$$

When $0 < \alpha \leqq 1$, 1.4.1 is an immediate consequence of the following elementary result.

LEMMA 1.4.2. *Suppose that* $0 < \alpha \leqq 1$ *and that* $f(x)$ *has bounded* α*-variation in some right-handed neighborhood of the point* $x = x_0$. *Then*

$$V_\alpha\{f(x); x_0 < x < x_0 + h\} = o(1)$$

as h approaches 0. *A similar result holds on the left.*

It follows from 1.4.2 that any function, with bounded α-variation over an open interval, has right- and left-handed limits at each point of the interval.

For $0 \leqq \alpha \leqq 1$, we say that $f(x)$ is in V_α over the interval $0 \leqq x \leqq 1$ if, given $\epsilon > 0$, there exists a $\delta > 0$ such that, for any set of disjoint intervals $0 \leqq x_1 < y_1 \leqq \cdots \leqq x_N < y_N \leqq 1$ for which

$$\left\{ \sum_{n=1}^{N} | y_n - x_n |^{1/\alpha} \right\}^{\alpha} < \delta,$$

we have

$$\left\{ \sum_{n=1}^{N} | f(y_n) - f(x_n) |^{1/\alpha} \right\}^{\alpha} < \epsilon.$$

V_0 is the class of functions continuous over $0 \leqq x \leqq 1$ and V_1 is the class of functions absolutely continuous over this interval.

The method of proof used in 1.2.2 gives us the following result.

THEOREM 1.4.3. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $f(x)$ *is in* V_α. *Then*

$$V_\alpha\{f(y); x \leqq y \leqq x + h\} = o(h^\alpha)$$

almost everywhere, and similarly on the left.

THEOREM 1.4.4 [6]. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $f(x)$ *is measurable and has period* 1. $f(x)$ *is in* V_α *if and only if*

$$V_\alpha\{f(x + h) - f(x); 0 \leqq x \leqq 1\} = o(1)$$

as h approaches 0.

We see that the class W_α includes V_α and, for $0 < \alpha < 1$, it is not difficult to show that the continuous function

$$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n \pi x$$

is in $W_\alpha - V_\alpha$. Hence a continuous function with bounded α-variation over $0 \leqq x \leqq 1$ is not necessarily in V_α. However, we can prove, for $0 \leqq \alpha \leqq 1$, that any continuous function in W_α which possesses a finite derivative everywhere in $0 \leqq x \leqq 1$, except perhaps on an enumerable set, is in V_α [2, Theorem 2.6].

1.5. Suppose that $f(x)$ and $g(x)$ are defined over the interval $0 \leqq x \leqq 1$ and that $g(x)$ has at most discontinuities of the 1st kind. The Stieltjes integral

1.5.1 $$\int_0^1 f(x) dg(x)$$

exists in the Young sense and is equal to I if, for $\epsilon > 0$, there exists a finite set

E, contained in $0 \leqq x \leqq 1$, such that, for any subdivision $0 = x_0 < \cdots < x_N = 1$ which contains E, we have

$$\Bigg| \sum_{n=1}^{N} f(\xi_n)\{g(x_n - 0) - g(x_{n-1} + 0)\} + \sum_{n=1}^{N-1} f(x_n)\{g(x_n + 0) - g(x_n - 0)\} + f(0)\{g(0+) - g(0)\} + f(1)\{g(1) - g(1 - 0)\} - I \Bigg| < \epsilon$$

for each set $x_0 < \xi_1 < x_1 < \cdots < x_{N-1} < \xi_N < x_N$. L. C. Young [12] has proved the following

THEOREM 1.5.2. *Suppose that $\alpha + \beta > 1$ and that $f(x)$ and $g(x)$ belong to W_α and W_β respectively. If $f(x)$ and $g(x)$ have no common discontinuities,* 1.5.1 *exists in the Riemann-Stieltjes sense. In any case,* 1.5.1 *exists in the Young sense.*

This theorem is not true in the limiting case when $\alpha + \beta = 1$. The following result is also an immediate consequence of Young's work.

THEOREM 1.5.3. *Suppose that $\alpha + \beta > 1$ and that $f(x)$ is continuous and in W_α. Suppose also that $\{g_n(x)\}$ is a sequence of uniformly bounded functions with uniformly bounded β-variation over $0 \leqq x \leqq 1$ which converges to $g(x)$, a function in W_β, on a set which includes the points $x = 0$ and $x = 1$ and which is dense in $0 \leqq x \leqq 1$. Then*

$$\lim_{n\to\infty} \int_0^1 f(x)dg_n(x) = \int_0^1 f(x)dg(x),$$

$$\lim_{n\to\infty} \int_0^1 g_n(x)df(x) = \int_0^1 g(x)df(x).$$

MOMENT PROBLEMS

2.1. In this chapter we study W_α, the class of functions with bounded α-variation over the interval $0 \leqq x \leqq 1$, by considering a moment problem. Suppose that $g(x)$ is any function in W_α for $0 < \alpha \leqq 1$. We call the numbers

$$\mu_n = \int_0^1 x^n dg(x), \qquad n = 0, 1, \cdots,$$

the Stieltjes moments of $g(x)$ and we say that $g(x)$ is normalized if

$$g(x) = \frac{1}{2}\{g(x + 0) + g(x - 0)\} \qquad \text{for } 0 < x < 1.$$

A uniqueness theorem follows from a known result [8, p. 60] after integration by parts.

THEOREM 2.1.1. *Suppose that* $0<\alpha\leqq 1$, *that* $g(x)$ *is normalized and in* W_α, *and that*

$$\int_0^1 x^n dg(x) = 0$$

for $n=0, 1, \cdots$. *Then* $g(x)$ *is identically constant in* $0\leqq x\leqq 1$.

For an arbitrary sequence of numbers, $\{\mu_n\}$, we define a linear functional over the space of polynomials by letting

$$\mu\{P\} = \mu\left\{\sum_{n=0}^{N} a_n x^n\right\} = \sum_{n=0}^{N} a_n\mu_n.$$

Following Hausdorff we define, for $k=0, 1, \cdots$ and $n=0, 1, \cdots, k$,

$$\lambda_{k,n}(x) = C_n^{k-n} x^n (1-x)^{k-n} \quad \text{and} \quad \lambda_{k,n} = \mu\{\lambda_{k,n}(x)\},$$

where C_s^r is the binomial coefficient

$$\binom{r+s}{r} = \frac{\Gamma(r+s+1)}{\Gamma(r+1)\Gamma(s+1)}\,.$$

THEOREM 2.1.2. *Suppose that* $0<\alpha\leqq 1$. *A necessary and sufficient condition that the set of numbers* $\{\mu_n\}$ *be the Stieltjes moments of a normalized function* $g(x)$ *in* W_α, *where* $V_\alpha\{g(x); 0\leqq x\leqq 1\}\leqq 1$, *is that*

$$\left\{\sum_{n=0}^{k} \left| \sum \lambda_{k,n}\right|^{1/\alpha}\right\}^{\alpha} \leqq 1$$

for all k.

Hausdorff [5] has proved this theorem for the case where $\alpha=1$.

2.2. In order to prove the sufficiency we derive two simple results.

LEMMA 2.2.1. *If* $\{\mu_n\}$ *is an arbitrary sequence of numbers, then*

$$\sum_{n=0}^{k}\left(\frac{n}{k}\right)^m \lambda_{k,n} = \mu_m + O\left(\frac{1}{k}\right)$$

for $m=0, 1, \cdots$.

Suppose that $f(x)$ is any function defined over the interval $0\leqq x\leqq 1$ and consider

$$B_k\{f; x\} = \sum_{n=0}^{k} f\left(\frac{n}{k}\right)\lambda_{k,n}(x),$$

the Bernstein polynomial of order k for $f(x)$. If $P_m(x)$ is a polynomial of degree m,

$$B_k\{P_m;\, x\} = P_m(x) + \sum_{\nu=1}^{m-1} \frac{Q_{m,\nu}(x)}{k^\nu},$$

where the polynomials $Q_{m,\nu}(x)$ do *not* depend on k and are identically zero when $P_m(x)$ is a constant [7, p. 8]. Setting $P_m(x)=x^m$, we have

$$\sum_{n=0}^{k} \left(\frac{n}{k}\right)^m \lambda_{k,n} = \mu\{B_k\{x^m;\, x\}\} = \mu_m + \sum_{\nu=1}^{m-1} \frac{\mu\{Q_{m,\nu}\}}{k^\nu},$$

and 2.2.1 follows.

LEMMA 2.2.2. *Suppose that* $0<\alpha\leqq 1$, *that* $\{g_k(x)\}$ *is a sequence of uniformly bounded functions, and that*

$$\liminf_{k\to\infty} V_\alpha\{g_k(x);\, 0 \leqq x \leqq 1\} \leqq 1.$$

There exists a function $g(x)$ *such that*

$$V_\alpha\{g(x);\, 0 \leqq x \leqq 1\} \leqq 1,$$

and a subsequence $\{k_j\}$ *such that* $g_{k_j}(x)\to g(x)$ *for every rational* x *in* $0\leqq x\leqq 1$, *including the end points* $x=0$ *and* $x=1$.

We can use the selection principle (or diagonal process) to define $g(x)$ on the rationals. If, for irrational x, we let

$$g(x) = \limsup_{r\to x} g(r), \qquad r \text{ rational},$$

then $g(x)$ satisfies the conditions of the lemma.

To prove the sufficiency part of 2.1.2, consider the step function $g_k(x)$ where $g_k(0)=0$ and

$$g_k(x) = \sum_{\nu=0}^{n} \lambda_{k,\nu} \qquad \text{for} \quad \frac{n}{k+1} < x \leqq \frac{n+1}{k+1}.$$

Since

$$|\, g_k(x)\,| \leqq V_\alpha\{g_k(x);\, 0 \leqq x \leqq 1\} \leqq 1,$$

there exists a function $g^*(x)$ such that

$$V_\alpha\{g^*(x);\, 0 \leqq x \leqq 1\} \leqq 1,$$

and a subsequence $\{k_j\}$ such that $g_{k_j}(x)\to g^*(x)$ for each rational x in $0\leqq x\leqq 1$. Let

2.2.3
$$g(0) = g^*(0), \qquad g(1) = g^*(1),$$
$$g(x) = \frac{1}{2}\{g^*(x+0) + g^*(x-0)\} \qquad \text{for } 0 < x < 1.$$

The function $g(x)$ is normalized and, from 1.5.3 and 2.2.1, we conclude that

$$\begin{aligned}\mu_m &= \lim_{k\to\infty} \sum_{n=0}^{k}\left(\frac{n}{k+1}\right)^m \lambda_{k,n} \\ &= \lim_{k\to\infty}\int_0^1 x^m dg_k(x) \\ &= \int_0^1 x^m dg^*(x) = \int_0^1 x^m dg(x).\end{aligned}$$

2.3. In order to prove the necessity we require a number of lemmas.

We say that a real function $f(x)$ is unimax in the interval $a\leqq x\leqq b$ if, for $a\leqq c<x<d\leqq b$, $f(x)\geqq \operatorname{Min}\{f(c), f(d)\}$.

LEMMA 2.3.1. *Suppose that* $0\leqq\alpha\leqq 1$ *and that* $X_n=\sum_{m=1}^{M} a_{n,m}x_m$ *for* $n=1,\cdots,N$, *where* $a_{n,m}$ *is a non-negative unimax function of* m *for each* n, *and where* $\sum_{n=1}^{N} a_{n,m}\leqq 1$ *for each* m. *Then*

$$\left\{\sum_{n=1}^{N}|X_n|^{1/\alpha}\right\}^{\alpha}\leqq\left\{\sum_{m=1}^{M}\left|\sum x_m\right|^{1/\alpha}\right\}^{\alpha}.$$

We consider 2.3.1 in a slightly different form.

LEMMA 2.3.2. *Suppose that* $0\leqq\alpha\leqq 1$ *and that* $X_n=\sum_{m=1}^{M} a_{n,m}x_m$ *for* $n=1,\cdots,N$, *where* $a_{n,m}$ *is a non-negative integer and is unimax as a function of* m *for each* n, *and where* $\sum_{n=1}^{N} a_{n,m}=R$ *for each* m. *Then*

$$\left\{\sum_{n=1}^{N}|X_n|^{1/\alpha}\right\}^{\alpha}\leqq R\left\{\sum_{m=1}^{M}\left|\sum x_m\right|^{1/\alpha}\right\}^{\alpha}.$$

LEMMA 2.3.3. *With the hypotheses of* 2.3.2 *we can write*

$$X_n=\sum_{r=1}^{R} y_{n,r}=\sum_{r=1}^{R}\left\{\sum_{m=1}^{M} a_{n,m,r}x_m\right\},$$

where $a_{n,m,r}$ *is either* 0 *or* 1 *and is unimax in* m *for each* n *and* r, *and where* $\sum_{n=1}^{N} a_{n,m,r}=1$ *for each* m *and* r.

Now 2.3.3 is true when $R=1$. Suppose it true for $R=k-1$ and consider the case where $R=k$.

A. There exists n_1 such that $a_{n_1,1}\geqq 1$. Let m_1 be the largest m for which $a_{n_1,m}\geqq 1$ and define

$$a_{n_1,m,1}=\begin{cases}1 & \text{for } 1\leqq m\leqq m_1,\\ 0 & \text{everywhere else.}\end{cases}$$

It is not difficult to see that $a_{n_1,m}-a_{n_1,m,1}$ is a non-negative integer and is unimax as a function of m.

B. If $m_1<M$, there exists $n_2\neq n_1$ such that $a_{n_2,m_1}<a_{n_2,m_1+1}$. Let m_2 be the largest m for which $a_{n_2,m}\geqq 1$ and define

$$a_{n_2,m,1} = \begin{cases} 1 & \text{for } m_1 < m \leqq m_2, \\ 0 & \text{everywhere else.} \end{cases}$$

Again $a_{n_2,m}-a_{n_2,m,1}$ is a non-negative integer and is unimax in m.

C. If $m_2<M$, we can find $n_3\neq n_2$, n_1 such that $a_{n_3,m_2}<a_{n_3,m_2+1}$, and we can define m_3 and the set $\{a_{n_3,m,1}\}$ for $m=1, \cdots, M$. After a finite number of steps we arrive at the place where $m_j=M$. We shall have defined a sequence of distinct integers, $n_1, n_2, \cdots, n_j$, and a set of coefficients, $\{a_{n,m,1}\}$, for $n=n_1, \cdots, n_j$ and $m=1, \cdots, M$. Let $a_{n,m,1}=0$ for $n\neq n_1, \cdots, n_j$ and $m=1, \cdots, M$.

The set $\{a_{n,m,1}\}$ satisfies the conditions in 2.3.3, the set $\{a_{n,m}-a_{n,m,1}\}$ satisfies the hypotheses of 2.3.2 with $R=k-1$, and 2.3.3 follows by applying the induction hypothesis to

$$X_n' = \sum_{m=1}^{M} \{a_{n,m} - a_{n,m,1}\} x_m.$$

For a fixed r, consider the set $\{y_{n,r}\}$. Since each element here is simply the sum of consecutive x_m, we see from 2.3.3 and Minkowski's inequality that

$$\begin{aligned} \left\{\sum_{n=1}^{N} | X_n |^{1/\alpha}\right\}^{\alpha} &= \left\{\sum_{n=1}^{N}\left| \sum_{r=1}^{R} y_{n,r}\right|^{1/\alpha}\right\}^{\alpha} \\ &\leqq \sum_{r=1}^{R}\left\{\sum_{n=1}^{N} | y_{n,r} |^{1/\alpha}\right\}^{\alpha} \\ &\leqq R\left\{\sum_{m=1}^{M} \left| \sum x_m \right|^{1/\alpha}\right\}^{\alpha}. \end{aligned}$$

Now 2.3.2 implies 2.3.1 when all the $a_{n,m}$ are rational and $\sum_{n=1}^{N} a_{n,m} = c_m = 1$. A simple limiting process removes the restriction that the $a_{n,m}$ be rational. When $0\leqq c_m\leqq 1$, we can apply our results to the set of linear forms

$$\sum_{m=1}^{M} a_{n,m}x_m, \qquad n = 1, \cdots, N,$$

$$(1 - c_m) x_m, \qquad m = 1, \cdots, M,$$

and show that

$$\left\{\sum_{n=1}^{N} | X_n |^{1/\alpha} + \sum_{m=1}^{M} | (1 - c_m) x_m |^{1/\alpha}\right\}^{\alpha} \leqq \left\{\sum_{m=1}^{M} \left| \sum x_m \right|^{1/\alpha}\right\}^{\alpha}.$$

This completes the proof for 2.3.1. We can assume that no x_m is zero. Hence when $\alpha>0$, we get strict inequality if, for any m, $c_m<1$.

LEMMA 2.3.4. *Suppose that $k=0, 1, \cdots$, and that $0\leqq m\leqq n\leqq k$. The function $f(x)=\sum_{\nu=m}^{n}\lambda_{k,\nu}(x)$ is unimax in $0\leqq x\leqq 1$.*

We can assume that $k\geqq 1$ and 2.3.4 follows immediately from the identity

2.3.5
$$\frac{d}{dx}\lambda_{k,n}(x) = \begin{cases} -k\lambda_{k-1,n}(x), & n=0, \\ k\{\lambda_{k-1,n-1}(x)-\lambda_{k-1,n}(x)\}, & 0<n<k, \\ k\lambda_{k-1,n-1}(x), & n=k. \end{cases}$$

To complete the proof for 2.1.2, fix k and consider any finite sequence $0=\nu_0<\cdots<\nu_N=k+1$. If

$$\theta_n = \sum_{\nu_{n-1}\leqq\nu<\nu_n}\lambda_{k,\nu} \quad\text{and}\quad \theta_n(x) = \sum_{\nu_{n-1}\leqq\nu<\nu_n}\lambda_{k,\nu}(x),$$

then

$$\theta_n = \int_0^1 \theta_n(x)dg(x)$$

and this integral exists in the Riemann-Stieltjes sense for each n. For each subdivision $0=y_0<\cdots<y_M=1$, let

$$X_n = \sum_{m=1}^{M}\theta_n(y_m)\{g(y_m)-g(y_{m-1})\}.$$

$\theta_n(y_m)$ is a non-negative unimax function of m and

$$\sum_{n=1}^{N}\theta_n(y_m) = \sum_{\nu=0}^{k}\lambda_{k,\nu}(y_m) = 1.$$

Applying 2.3.1 we get

$$\left\{\sum_{n=1}^{N}|X_n|^{1/\alpha}\right\}^{\alpha} \leqq \left\{\sum_{m=1}^{M}|\sum g(y_m)-g(y_{m-1})|^{1/\alpha}\right\}^{\alpha}$$

which completes the argument.

2.4. We have discussed the Stieltjes moments of a function $g(x)$, i.e. the sequence $\{\mu_n\}$ where

$$\mu_n = \int_0^1 x^n dg(x), \qquad n=0, 1, \cdots.$$

We can also consider the moment sequence

$$\mu_n = \int_0^1 x^n g(x)dx, \qquad n=0, 1, \cdots,$$

where the integral is interpreted in the Lebesgue sense. We call such a sequence of numbers the Lebesgue moments of $g(x)$ and we have the following

THEOREM 2.4.1. *Suppose that* $0<\alpha\leqq 1$. *A necessary and sufficient condition that the set of numbers* $\{\mu_n\}$ *be the Lebesgue moments of a normalized function* $g(x)$ *in* W_α, *where*

$$V_\alpha\{g(x);\, 0 \leqq x \leqq 1\} \leqq 1,$$

is that

$$(k+1)\left\{\sum_{n=1}^{k} \left| \sum \lambda_{k,n} - \lambda_{k,n-1} \right|^{1/\alpha}\right\}^{\alpha} \leqq 1$$

for all k.

The necessity follows immediately from 2.1.2 since, with the help of 2.3.5, we can write

$$\begin{aligned}(k+1)\{\lambda_{k,n} - \lambda_{k,n-1}\} &= \int_0^1 (k+1)\{\lambda_{k,n}(x) - \lambda_{k,n-1}(x)\}g(x)dx \\ &= \int_0^1 \lambda_{k+1,n}(x)dg(x)\end{aligned}$$

for $0<n\leqq k$.

For the sufficiency, observe that

2.4.2 $$\left| \mu_0 - (k+1)\lambda_{k,m} \right| \leqq \sum_{n=0}^{k} \left| \lambda_{k,n} - \lambda_{k,m} \right| \leqq 1$$

for $0\leqq m\leqq k$. Consider the step function $g_k(x)$ where $g_k(0)=(k+1)\lambda_{k,0}$ and

$$g_k(x) = (k+1)\lambda_{k,n} \qquad \text{for } \frac{n}{k+1} < x \leqq \frac{n+1}{k+1}.$$

From 2.4.2 and 2.2.2 it follows that there exists a function $g^*(x)$ such that

$$V_\alpha\{g^*(x);\, 0 \leqq x \leqq 1\} \leqq 1,$$

and a subsequence $\{k_j\}$ such that $g_{k_j}(x)\to g^*(x)$ for each rational x in $0\leqq x\leqq 1$. Define $g(x)$ as in 2.2.3. From 1.5.3, 2.2.1, and 2.4.2 we can conclude that

$$\begin{aligned}\mu_m &= \lim_{k\to\infty} \sum_{n=0}^{k} \left(\frac{n}{k+1}\right)^m \lambda_{k,n} \\ &= \lim_{k\to\infty} \int_0^1 x^m g_k(x)dx \\ &= \int_0^1 x^m g^*(x)dx = \int_0^1 x^m g(x)dx.\end{aligned}$$

2.5. Turning back to 2.3.1, we deduce two alternative forms for this inequality which are useful in later work.

LEMMA 2.5.1. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $Y_n = \sum_{m=1}^{M} b_{n,m}(y_m - y_0)$ *for* $n = 0, \cdots, N$, *where* $b_{n,m}$ *is non-negative for all* m *and* n, *where* $\sum_{m=1}^{M} b_{n,m}$ *is nondecreasing in* n *and bounded by* 1, *and where, for each* $0 \leqq n' < n'' \leqq N$, $b_{n'',m} - b_{n',m}$ *is at first nonpositive and then non-negative as* m *increases from* 1 *to* M. *Then*

$$\left\{ \sum_{n=1}^{N} \left| \sum Y_n - Y_{n-1} \right|^{1/\alpha} \right\}^{\alpha} \leqq \left\{ \sum_{m=1}^{M} \left| \sum y_m - y_{m-1} \right|^{1/\alpha} \right\}^{\alpha}.$$

LEMMA 2.5.2. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $Y_n = \sum_{m=0}^{M} b_{n,m} y_m$ *for* $n = 0, \cdots, N$, *where* $b_{n,m}$ *is non-negative for all* m *and* n, *where* $\sum_{m=0}^{M} b_{n,m} = 1$, *and where, for each* $0 \leqq n' < n'' \leqq N$, $b_{n'',m} - b_{n',m}$ *is at first nonpositive and then non-negative as* m *increases from* 0 *to* M. *Then*

$$\left\{ \sum_{n=1}^{N} \left| \sum Y_n - Y_{n-1} \right|^{1/\alpha} \right\}^{\alpha} \leqq \left\{ \sum_{m=1}^{M} \left| \sum y_m - y_{m-1} \right|^{1/\alpha} \right\}^{\alpha}.$$

For 2.5.1, let $x_m = y_m - y_{m-1}$ and pick any sequence of integers, $0 = k_0 < \cdots < k_{N'} = N$. If

$$X_n = Y_{k_n} - Y_{k_{n-1}} = \sum_{m=1}^{M} a_{n,m} x_m,$$

then

$$a_{n,m} = \sum_{\mu=m}^{M} \left\{ b_{k_n,\mu} - b_{k_{n-1},\mu} \right\}.$$

The difference $a_{n,m+1} - a_{n,m} = -\left\{ b_{k_n,m} - b_{k_{n-1},m} \right\}$ is at first non-negative and then nonpositive as m increases; hence $a_{n,m}$ is unimax in m. We see that

$$a_{n,1} = \sum_{m=1}^{M} b_{k_n,m} - \sum_{m=1}^{M} b_{k_{n-1},m} \geqq 0,$$

$$a_{n,M} = b_{k_n,M} - b_{k_{n-1},M} \geqq 0,$$

and the unimax property ensures that $a_{n,m}$ is non-negative for all m and n. Finally

$$\sum_{n=1}^{N'} a_{n,m} = \sum_{\mu=m}^{M} \left\{ b_{N,\mu} - b_{0,\mu} \right\} \leqq 1,$$

and we can apply 2.3.1.

For 2.5.2 observe that the set $\{b_{n,m}\}$, for $n = 0, \cdots, N$ and $m = 0, \cdots, M$, satisfies the hypotheses of 2.5.1. Since the sums

$$\sum_{m=0}^{M} b_{n,m}(y_m - y_0) \quad \text{and} \quad \sum_{m=0}^{M} b_{n,m}y_m$$

differ by a constant which is independent of n, the conclusion follows immediately.

From 2.5.2 we can deduce the following result concerning Bernstein polynomials.

THEOREM 2.5.3. *If* $0 \leqq \alpha \leqq 1$,

$$V_\alpha\{B_k\{f;\, x\};\, 0 \leqq x \leqq 1\} \leqq \left\{\sum_{m=1}^{k} \left|\sum f\left(\frac{m}{k}\right) - f\left(\frac{m-1}{k}\right)\right|^{1/\alpha}\right\}^{\alpha}.$$

Let $0 < x_0 < \cdots < x_N < 1$ be any subdivision of the interval $0 < x < 1$. We see that

$$B_k\{f;\, x_n\} = \sum_{m=0}^{k} f\left(\frac{m}{k}\right)\lambda_{k,m}(x_n)$$

where $\lambda_{k,m}(x_n)$ is non-negative for all m and n, and where $\sum_{m=0}^{k} \lambda_{k,m}(x_n) = 1$. For $0 \leqq n' < n'' \leqq N$,

$$\lambda_{k,m}(x_{n''}) - \lambda_{k,m}(x_{n'}) = \lambda_{k,m}(x_{n''})\left\{1 - \left(\frac{x_{n'}}{x_{n''}}\right)^m \left(\frac{1 - x_{n'}}{1 - x_{n''}}\right)^{k-m}\right\}.$$

Since $0 < x_{n'} < x_{n''} < 1$, the bracketed quantity is negative for $m = 0$, positive for $m = k$, and strictly increasing in m. Applying 2.5.2 completes the proof.

In conclusion we add that 2.3.1, 2.5.1, and 2.5.2 are valid when M and/or $N = \infty$.

A FALTUNG THEOREM

3.1. In one of his papers [11], L. C. Young considered a Stieltjes Faltung of the form

3.1.1 $$s(x) = \int_0^1 f(x,\, y)dg(y).$$

We present here a theorem suggested by Young's results.

Suppose that $f(x, y)$ and $g(y)$ have period 1 in y and define $F(x, y)$ as the integral $\int_0^1 f(x, y+t)g(t)dt$ which we assume exists in the Lebesgue sense for all $0 \leqq x, y \leqq 1$. Let

3.1.2 $$\begin{aligned} s_n(x) &= 2^n\{F(x,\, 0) - F(x,\, 2^{-n})\} \\ &= 2^n \int_0^1 f(x,\, t)\{g(t) - g(t - 2^{-n})\}dt \end{aligned}$$

for $n = 0, 1, \cdots$. Then $s_0(x) \equiv 0$ and the following is easily verified.

LEMMA 3.1.3. *Suppose that $g(y)$ is continuous and that the integral* 3.1.1 *exists in the Riemann-Stieltjes sense for $x=x_0$. Then $s(x_0)=\lim_{n\to\infty} s_n(x_0)$ and, for $n\geqq 1$, we have*

$$s_n(x) - s_{n-1}(x) = 2^{n-1}\int_0^1 \{f(x, t) - f(x, t + 2^{-n})\}\{g(t) - g(t - 2^{-n})\}dt.$$

Our principal result is as follows.

THEOREM 3.2. (*Cf.* [11, *Theorem* 6.1].) *Suppose that $0<\alpha, \beta, \gamma\leqq 1$, $0<\lambda=\beta+\gamma-1$, and $\mu=\alpha\lambda/\beta$. Suppose also that $f(x, y)$ and $g(y)$ have period* 1 *in y, that $g(y)$ is continuous, and that*

3.2.1 $$V_\alpha\{f(x, y); 0 \leqq x \leqq 1\} \leqq A \qquad \text{for each } y,$$

3.2.2 $$V_\beta\{f(x, y); 0 \leqq y \leqq 1\} \leqq B \qquad \text{for each } x,$$

3.2.3 $$V_\gamma\{g(y); 0 \leqq y \leqq 1\} \leqq C.$$

If $s(x)$ is the Stieltjes Faltung 3.1.1, *then*

3.2.4 $$V_\mu\{s(x); 0 \leqq x \leqq 1\} \leqq k(\lambda)A^{\lambda/\beta}B^{1-\lambda/\beta}C,$$

where $k(\lambda)$ is a finite constant.

If $B=0$ and/or $C=0$, $s(x)\equiv 0$ and 3.2.4 follows immediately. Hence we can assume that $B=C=1$.

Obviously we can suppose that $f(x, y)$ and $g(y)$ are real. By an argument similar to that used in 1.2.1, we can find a strictly increasing continuous function $\phi(t)$ such that

$$\phi(0)=0 \text{ and } \phi(t+1)=\phi(t)+1$$

for all t, and such that

$$| g\{\phi(t'')\} - g\{\phi(t')\} | < 2(t'' - t')^\alpha$$

for each $t'<t''$. Furthermore, for each $0\leqq x\leqq 1$,

$$V_\beta\{f(x, \phi(t)); 0 \leqq t \leqq 1\} = V_\beta\{f(x, y); 0 \leqq y \leqq 1\},$$

$$\int_0^1 f\{x, \phi(t)\}dg\{\phi(t)\} = \int_0^1 f(x, y)dg(y),$$

and, by performing a change of variable, we replace condition 3.2.3 by the condition

3.2.5 $$| g(y'') - g(y') | < 2(y'' - y')^\alpha$$

for each $y'<y''$.

Since $g(y)$ is continuous and $\beta+\gamma>1$, the Faltung $s(x)$ exists in the Riemann-Stieltjes sense for each x and is equal to $\lim_{n\to\infty} s_n(x)$. Using 3.1.3, 3.2.5, Jensen's inequality, 3.2.2, and 1.3.4 we have

$$
\begin{aligned}
|s_n(x) - s_{n-1}(x)| &\leqq 2^{n-1}\int_0^1 |f(x, t + 2^{-n}) - f(x, t)|\,|g(t) - g(t - 2^{-n})|\,dt \\
&\leqq 2^{n(1-\gamma)}\left\{\int_0^1 |f(x, t + 2^{-n}) - f(x, t)|^{1/\beta} dt\right\}^{\beta} \\
&\leqq 2^{\beta}\cdot 2^{-n\lambda},
\end{aligned}
$$

and summing on n we get

3.2.6
$$|s(x) - s_n(x)| \leqq 2^{\beta}\sum_{\nu=n+1}^{\infty} 2^{-\nu\lambda} = c(\lambda)2^{-n\lambda}$$

for $n=0, 1, \cdots$. Fix n and consider $0 \leqq x' < x'' \leqq 1$. From 3.1.2, 3.2.5, and Jensen's inequality we have

3.2.7
$$
\begin{aligned}
|s_n(x'') - s_n(x')| &\leqq 2^{n}\int_0^1 |f(x'', t) - f(x', t)|\,|g(t) - g(t - 2^{-n})|\,dt \\
&\leqq 2^{n(1-\gamma)}\cdot 2\Delta
\end{aligned}
$$

where

$$\Delta = \left\{\int_0^1 |f(x'', t) - f(x', t)|^{1/\alpha} dt\right\}^{\alpha}.$$

By considering three different cases we prove that

3.2.8
$$|s(x'') - s(x')| \leqq k(\lambda)\Delta^{\mu/\alpha},$$

where $k(\lambda)$ is a finite constant.

A. Suppose that $1 < \Delta < \infty$. Setting $n=0$ in 3.2.6 gives us 3.2.8 if we choose $k(\lambda) \geqq 2c(\lambda)$.

B. Suppose that $0 < \Delta \leqq 1$. Choose $n \geqq 1$ so that $2^{-n\beta} < \Delta \leqq 2^{-(n-1)\beta}$, and with 3.2.6 we have

$$|s(x) - s_n(x)| < c(\lambda)\Delta^{\mu/\alpha}$$

for $0 \leqq x \leqq 1$. From 3.2.7 we get

$$|s_n(x'') - s_n(x')| \leqq 4\Delta^{\mu/\alpha},$$

and 3.2.8 follows if we choose $k(\lambda) \geqq 2(\lambda) + 4$.

C. Suppose that $\Delta = 0$. We see from 3.1.3 and 3.2.7 that

$$|s(x'') - s(x')| = \lim_{n\to\infty} |s_n(x'') - s_n(x')| = 0,$$

and 3.2.8 follows if we choose $k(\lambda) \geqq 0$.

To complete the proof for 3.2, take any subdivision $0 = x_0 < \cdots < x_N = 1$. From 3.2.8 we have

$$\left\{\sum_{n=1}^{N} | s(x_n) - s(x_{n-1}) |^{1/\mu}\right\}^{\mu} \leqq k(\lambda)\left\{\int_0^1 \sum_{n=1}^{N} | f(x_n, t) - f(x_{n-1}, t) |^{1/\alpha} dt\right\}^{\mu}$$
$$\leqq k(\lambda)A^{\lambda/\beta}.$$

The proof introduces unnecessary restrictions. For example, Young's argument [11, p. 459] allows us to consider the case where $g(y)$ is not continuous. We conclude this chapter by simply stating the following generalization of 3.2.

THEOREM 3.3. *Suppose that* $0<\alpha, \beta, \gamma \leqq 1$, $0<\lambda=\beta+\gamma-1$, *and* $u=\alpha\lambda/\beta$. *Suppose also that* 3.2.1, 3.2.2, *and* 3.2.3 *hold. If* $s(x)$ *is the Stieltjes Faltung* 3.1.1, *then*

$$V_\mu\{s(x) - \delta f(x, 0); 0 \leqq x \leqq 1\} \leqq k(\lambda)A^{\lambda/\beta}B^{1-\lambda/\beta}C,$$

where $\delta=g(1)-g(0)$ *and* $k(\lambda)$ *is a finite constant.*

APPLICATIONS TO INFINITE SERIES

4.1. In this chapter we apply the notion of α-variation to the study of infinite series. We say that the series

4.1.1 $$\sum_{n=0}^{\infty} a_n$$

is α-convergent if, given $\epsilon>0$, there exists $N(\epsilon)$ such that

$$\left\{\sum_{\nu=m}^{n} | \sum a_\nu |^{1/\alpha}\right\}^{\alpha} < \epsilon$$

for $N(\epsilon) \leqq m < n$. 0-convergence is ordinary convergence and 1-convergence is absolute convergence. If $0 \leqq \alpha < \beta \leqq 1$, we have by Jensen's inequality

$$\left\{\sum_{\nu=m}^{n} | \sum a_\nu |^{1/\alpha}\right\}^{\alpha} \leqq \left\{\sum_{\nu=m}^{n} | \sum a_\nu |^{1/\beta}\right\}^{\beta},$$

and thus a β-convergent series is always α-convergent.

We can extend the notion of α-convergence to sequences. We call $\{S_n\}$ an α-convergent sequence if S_n is the nth partial sum of an α-convergent series. From Minkowski's inequality we see that any finite linear combination of α-convergent series (sequences) is itself an α-convergent series (sequence). We also have the following result.

LEMMA 4.1.2. (*Cf. Lemma* 1.4.2.) *Suppose that* $0<\alpha\leqq 1$. *The series* 4.1.1 *is α-convergent if and only if*

$$\left\{\sum_{n=0}^{\infty} | \sum a_n |^{1/\alpha}\right\}^{\alpha} < \infty.$$

The same type of result is true for sequences.

Any series derived from a 1-convergent series by a rearrangement of terms is convergent to the sum of the original series. However, when $\alpha<1$, an α-convergent series is "conditionally convergent" and little can be said about rearrangement.

A second important property of 1-convergent series is found in multiplication theorems. For example it is well known that the Cauchy product of a 1-convergent series by a 0-convergent series is 0-convergent to the product of the sums of the series. We have the following extension of this result.

THEOREM 4.1.3. *Suppose that $0<\alpha, \beta\leqq 1$, and that $0<\gamma=\alpha+\beta-1$. Then the Cauchy product of an α-convergent series by a β-convergent series is γ-convergent to the product of the sums.*

This theorem follows easily from the following specialization of 3.3.

THEOREM 4.1.4. *Suppose that $0<\alpha$, $\beta\leqq 1$, and that $0<\gamma=\alpha+\beta-1$. If $f(x)$ has bounded α-variation over $0\leqq x<\infty$ and if $g(x)$ has bounded β-variation over $0\leqq x<\infty$, then the Stieltjes Faltung*

$$s(x) = \int_0^x f(x-y)dg(y)$$

exists in the Young sense for each x and has bounded γ-variation over $0\leqq x<\infty$.

Theorem 4.1.3 holds in the limiting case where $\alpha+\beta=1$ if and only if $\alpha=0$ or 1; 4.1.3 is also true when one considers the more general Dirichlet product [3, p. 239] instead of the Cauchy product.

4.2. We can apply our scale to the study of Cesaro and Abel summability. Suppose that S_n^k is the nth Cesaro mean of order k for the series 4.1.1. We say that 4.1.1 is summable $(C, k; \alpha)$ to S if the sequence $\{S_n^k\}$ is α-convergent to S. Thus $(C, k; 0)$ summability is ordinary Cesaro summability and $(C, k; 1)$ summability is absolute Cesaro summability. Consider the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and assume that this series converges for $0\leqq x<1$. We say that 4.1.1 is summable $(A; \alpha)$ to S if $f(x)$ has bounded α-variation over $0\leqq x<1$ and if $\lim_{x\to 1-} f(x)=S$. $(A; 0)$ summability is ordinary Abel summability and $(A; 1)$ summability is absolute Abel summability. (See [1, p. 11] for references on absolute summability.) A linear combination of series, summable $(C, k; \alpha)$ for some k and $0\leqq\alpha\leqq 1$, is itself a series summable $(C, k; \alpha)$ and similarly for the $(A; \alpha)$ method.

When $0\leqq\alpha<\beta\leqq 1$, a series summable $(C, k; \beta)$ to S is summable $(C, k; \alpha)$ to S and a series summable $(A; \beta)$ to some limit is summable $(A; \alpha)$ to the same limit. We can also establish the following consistency result.

THEOREM 4.2.1. *Suppose that* $0\leqq\alpha\leqq 1$ *and that* $k>j>-1$. *A series summable* $(C, j;\alpha)$ *to* S *is summable* $(C, k;\alpha)$ *and* $(A;\alpha)$ *to* S. *If* S_n^j *and* S_n^k *are the nth Cesaro means of order j and k respectively for* 4.1.1, *we have*

4.2.2 $$\left\{\sum_{n=1}^{\infty}\left|\sum S_n^k - S_{n-1}^k\right|^{1/\alpha}\right\}^{\alpha} \leqq \left\{\sum_{n=1}^{\infty}\left|\sum S_n^j - S_{n-1}^j\right|^{1/\alpha}\right\}^{\alpha},$$

4.2.3 $$V_\alpha\left\{\sum_{n=0}^{\infty} a_n x^n;\ 0\leqq x\leqq 1\right\} \leqq \left\{\sum_{n=1}^{\infty}\left|\sum S_n^k - S_{n-1}^k\right|^{1/\alpha}\right\}^{\alpha}.$$

For the first of these inequalities let $k-j=i>0$ and write

$$S_n^k = \sum_{m=0}^{n}\frac{C_{n-m}^{i-1}C_m^j}{C_n^k}S_m^j = \sum_{m=0}^{\infty} b_{n,m}S_m^j,$$

where C_s^r is the binomial coefficient

$$\binom{r+s}{r}.$$

$b_{n,m}$ is non-negative for all m and n and

$$\sum_{m=0}^{\infty} b_{n,m} = \frac{1}{C_n^k}\sum_{m=0}^{n} C_{n-m}^{i-1}C_m^j = 1.$$

When $0\leqq n'<n''<\infty$,

$$b_{n'',m} - b_{n',m} = b_{n',m}\left\{\frac{(n''-m+i-1)\cdots(n'-m+i)}{(n''-m)\cdots(n'-m+1)}\cdot\frac{n''\cdots(n'+1)}{(n''+k)\cdots(n'+k+1)} - 1\right\}$$

for $0\leqq m\leqq n'$. If $0<i<1$, the bracketed quantity is negative for $0\leqq m\leqq n'$ and, since $b_{n',m}=0$ for $m>n'$, we can apply 2.5.2.

For any subdivision $0<x_0<\cdots<x_N<1$ let

$$\sum_{m=0}^{\infty} a_m(x_n)^m = \sum_{m=0}^{\infty} C_m^k(1-x_n)^{k+1}(x_n)^m S_m^k = \sum_{m=0}^{\infty} b_{n,m}S_m^k.$$

Again $b_{n,m}$ is non-negative for all m and n,

$$\sum_{m=0}^{\infty} b_{n,m} = (1-x_n)^{k+1}\sum_{m=0}^{\infty} C_m^k(x_n)^m = 1,$$

and, for $0\leqq n'<n''\leqq N$,

$$b_{n'',m} - b_{n',m} = b_{n',m}\left\{\left(\frac{x_{n''}}{x_{n'}}\right)^m\left(\frac{1 - x_{n''}}{1 - x_{n'}}\right)^{k+1} - 1\right\}.$$

The bracketed factor here is an increasing function of m and 4.2.3 follows from 2.5.2.

The rest of the theorem follows from these two inequalities and a classical result.

4.3. The direct converse of 4.2.1 is not true. The coefficients in the power series expansion for

$$f(x) = e^{1/(1+x)}$$

constitute a series which is summable $(A;\ 1)$ but which is not summable $(C,\ k;\ 0)$ for any finite k. However, we can prove some "corrected converses" and the following theorems extend two results due to A. Tauber.

LEMMA 4.3.1. 1. $\alpha_n = \sum_{m=1}^{n-1} (1-(1-1/n)^m)/m$ *is a positive, increasing sequence bounded by* 1 *for* $n \geqq 2$.

2. $\beta_n = (1-(1-1/n)^n)/n$ *is a positive, decreasing sequence bounded by* 1 *for* $n \geqq 1$.

3. $\gamma_n = \sum_{m=n+1}^{\infty} ((1-1/n)^m)/m$ *is a positive, increasing sequence bounded by* 1 *for* $n \geqq 1$.

Consider the first sequence. If $r>1$, $x^r - y^r > ry^{r-1}(x-y)$ for any two positive and unequal x and y. Hence for $n \geqq 2$ we have

$$\begin{aligned}
\alpha_{n+1} - \alpha_n &= \sum_{m=1}^{n-1} \frac{(1 - 1/n)^m - (1 - 1/(n+1))^m}{m} + \frac{1 - (1 - 1/(n+1))^n}{n} \\
&\geqq -\frac{1}{n(n+1)} \sum_{m=0}^{n-2}\left(1 - \frac{1}{n+1}\right)^m + \frac{1 - (1 - 1/(n+1))^n}{n} \\
&\geqq \frac{1}{n}\left\{\left(1 - \frac{1}{n+1}\right)^{n-1} - \left(1 - \frac{1}{n+1}\right)^n\right\} > 0.
\end{aligned}$$

For the boundedness we see that

$$\alpha_n = \sum_{m=1}^{n-1} \frac{1 - (1 - 1/n)^m}{m} \leqq \frac{n-1}{n} < 1.$$

The proofs of 4.3.1.2 and 4.3.1.3 follow along similar lines.

We can now generalize Tauber's first theorem.

THEOREM 4.3.2. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* 4.1.1 *is summable* $(A;\ \alpha)$ *to* S. *If the sequence* $\{na_n\}$ *is* α*-convergent to* 0, 4.1.1 *is* α*-convergent to* S.

We can assume that $0 < \alpha \leqq 1$. For $0 \leqq x < 1$ let

$$f(x) = \sum_{m=0}^{\infty} a_m x^m$$

and let S_n be the nth partial sum for 4.1.1, i.e.

$$S_n = \sum_{\nu=0}^{n} a_\nu.$$

For $n \geqq 2$ we can write

$$S_n - f\left(1 - \frac{1}{n}\right) = A_n + B_n - C_n,$$

where

$$A_n = \sum_{m=1}^{n-1} \frac{1 - (1 - 1/n)^m}{m} (ma_m) = \sum_{m=1}^{\infty} \alpha_{n,m}(ma_m),$$

$$B_n = \frac{1 - (1 - 1/n)^n}{n} (na_n) = \beta_n(na_n),$$

$$C_n = \sum_{m=n+1}^{\infty} \frac{(1 - 1/n)^m}{m} (ma_m) = \sum_{m=1}^{\infty} \gamma_{n,m}(ma_m).$$

$\alpha_{n,m}$ and $\gamma_{n,m}$ are non-negative for all m and n and, by 4.3.1, both $\sum_{m=1}^{\infty} \alpha_{n,m}$ and $\sum_{m=1}^{\infty} \gamma_{n,m}$ are increasing in n and bounded by 1. These two sets of coefficients satisfy the hypotheses of 2.5.1. Hence

$$\left\{\sum_{n=3}^{\infty} \left| \sum A_n - A_{n-1} \right|^{1/\alpha}\right\}^{\alpha} \quad \text{and} \quad \left\{\sum_{n=3}^{\infty} \left| \sum C_n - C_{n-1} \right|^{1/\alpha}\right\}^{\alpha}$$

are majorized by

$$\left\{\sum_{m=1}^{\infty} \left| \sum ma_m - (m-1)a_{m-1} \right|^{1/\alpha}\right\}^{\alpha},$$

and $\{A_n\}$ and $\{C_n\}$ are α-convergent sequences. The sequence $\{B_n\}$ is also α-convergent since $\{\beta_n\}$ is 1-convergent to 0. Because 4.1.1 is $(A; \alpha)$ summable, the sequence $\{f(1-1/n)\}$ is α-convergent, and we conclude that $\{S_n\}$ is α-convergent to S.

THEOREM 4.3.3. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that the series* 4.1.1 *is summable* $(A; \alpha)$ *to* S. 4.1.1 *is* α*-convergent to* S *if and only if the sequence*

$$\left\{\frac{a_1 + \cdots + na_n}{n}\right\}$$

is α*-convergent to* 0.

For the necessity let S_n^1 be the nth Cesaro mean of order 1 for 4.1.1 and we see from 4.2.1 that the sequence

$$S_n - S_{n-1}^1 = \frac{a_1 + \cdots + na_n}{n}$$

is α-convergent to 0.

For the sufficiency set $b_0 = 0$ and let

$$\sum_{m=0}^{n} b_m = \frac{a_1 + \cdots + na_n}{n}$$

for $n \geqq 1$. If $a_n = b_n + c_n$ for all n, then

$$c_n = \frac{a_1 + \cdots + (n-1)a_{n-1}}{n(n-1)}$$

for $n \geqq 2$. $\sum_{n=0}^{\infty} a_n$ is $(A; \alpha)$ summable to S, $\sum_{n=0}^{\infty} b_n$ is α-convergent to 0, and hence the series $\sum_{n=0}^{\infty} c_n$ is also $(A; \alpha)$ summable to S. By 4.3.2 we see that this series is then α-convergent to S and this completes the proof.

We saw in 4.3.2 how the Tauberian condition

$$\{na_n\} \text{ is } \alpha\text{-convergent to } 0$$

allowed us to pass from summability $(A; \alpha)$ to summability $(C, 0; \alpha)$ or α-convergence. Actually more is true and we have the following result.

THEOREM 4.3.4. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* 4.1.1 *is α-convergent. If the sequence* $\{na_n\}$ *is α-convergent to* 0, 4.1.1 *is summable* $(C, k; \alpha)$ *to its sum for every* $k > -1$.

Pick $\delta > 0$. Let S_n^δ and $S_n^{\delta-1}$ be the nth Cesaro means of order δ and $\delta - 1$ respectively for 4.1.1, and let T_n^δ be the nth Cesaro mean of order δ for the series whose nth partial sum is na_n. From the identity $(n+\delta)C_{n-\nu}^{\delta-1} - \delta C_{n-\nu}^{\delta} = \nu C_{n-\nu}^{\delta-1}$, we see that

$$S_n^{\delta-1} - S_n^{\delta} = \frac{1}{\delta} T_n^{\delta}.$$

By 4.2.1, $\{S_n^\delta\}$ and $\{T_n^\delta\}$ are α-convergent to S and 0 respectively and thus $\{S_n^{\delta-1}\}$ is α-convergent to S.

In conclusion we construct, for $0 \leqq \alpha < 1$, a series which is summable $(C, k; \alpha)$ for every $k > -1$ and which is not summable $(A; \beta)$ for any $\beta > \alpha$.

Let $\{b_k\}$ be any positive decreasing sequence of numbers which approach zero such that

$$\left\{ \sum_{k=1}^{\infty} b_k^{1/\alpha} \right\}^{\alpha} < \infty \quad \text{and} \quad \left\{ \sum_{k=1}^{\infty} b_k^{1/\beta} \right\}^{\beta} = \infty$$

for each $\beta > \alpha$. Define a sequence of integers, $1 = n_0 < n_1 < \cdots$, and a set of positive numbers, $\{c_k\}$, such that

$$\sum_{n_{k-1} \leqq n < n_k} \frac{1}{n} \geqq 2^k b_k = c_k \sum_{n_{k-1} \leqq n < n_k} \frac{1}{n}$$

for $k = 1, 2, \cdots$. Set $a_0 = 0$ and $a_n = ((-1)^k/n)2^{-k} c_k$ for $n_{k-1} \leqq n < n_k$. Then

$$\left\{ \sum_{n=n_k}^{\infty} \left| \sum a_n \right|^{1/\rho} \right\} = \left\{ \sum_{j=k+1}^{\infty} b_j^{1/\rho} \right\}^{\rho}$$

for each $0 \leqq \rho \leqq 1$, and the series is α-convergent but not β-convergent for any $\beta > \alpha$. Since

$$\sum_{n=1}^{\infty} \left| na_n - (n-1)a_n \right| = 2 \sum_{k=1}^{\infty} 2^{-k} c_k < 2,$$

the sequence $\{na_n\}$ is 1-convergent to 0, and we see from 4.3.2 and 4.3.4 that this series has the desired properties.

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